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The compatibility of the filtration of mapping class groups of two surfaces pasted along the boundaries

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Abstract

Let Σ_g^n be an orientable surface of genus $g \geqslant 0$ with $n \geqslant 0$ punctures and Γ_g^n be its pure mapping class group. The group Γ_g^n has a filtration $\{\Gamma_g^n[m]\}_{m\geqslant 1}$ induced from its action on the fundamental group of Σ_g^n. If $g = g_1 + g_2$ and $n = n_1 + n_2 - 2$, we have a homomorphism $\Gamma_{g_1}^{n_1}[3] \times \Gamma_{g_2}^{n_2}[3] \to \Gamma_g^n$ [Math. Ann. 304 (1996) 99], which is induced from pasting two surfaces with one boundary component along their boundaries. That this homomorphism preserves the filtration strictly has been shown by Nakamura in the case that $n \geqslant 1$. We shall show that this holds also in the case that $n = 0$. As an application, we obtain a lower bound of the rank of the graded module associated with the filtration of $\Gamma_g (= \Gamma_g^0)$. © 2000 Elsevier Science B.V. All rights reserved.

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Introduction

Let Σ_g^n be an orientable surface of genus $g \geqslant 0$ with $n \geqslant 0$ punctures and Γ_g^n be its pure mapping class group. The fundamental group $\pi_1(\Sigma_g^n)$ of Σ_g^n has a canonical filtration called the weight filtration (due to Oda and Kaneko), and this induces naturally a filtration

$$\Gamma_g^n = \Gamma_g^n[0] \supset \Gamma_g^n[1] \supset \Gamma_g^n[2] \supset \cdots \supset \Gamma_g^n[m] \supset \Gamma_g^n[m+1] \cdots$$

on the group Γ_g^n. The aim of this paper is to give some results on this filtration.

Let $\Sigma_{g_1,1}^{n_1-1}$ (respectively $\Sigma_{g_2,1}^{n_2-1}$) be an orientable surface of genus g_1 (respectively g_2) with $(n_1 - 1)$ (respectively $(n_2 - 1)$) punctures and one boundary component. Let Σ_g^n

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be the surface of genus $g(=g_1+g_2)$ with $n(=n_1+n_2-2)$ punctures obtained from $\Sigma_{g_1,1}^{n_1-1}$ and $\Sigma_{g_2,1}^{n_2-1}$ by pasting their boundaries. Let $\Gamma_{g_1,1}^{n_1-1}$ (respectively $\Gamma_{g_2,1}^{n_2-1}$) be the pure mapping class group of $\Sigma_{g_1,1}^{n_1-1}$ (respectively $\Sigma_{g_2,1}^{n_2-1}$). Then, a pair of mapping classes (σ_1,σ_2) $(\sigma_i\in\Gamma_{g_i,1}^{n_i-1})$ defines a mapping class of Σ_g^n and we have a homomorphism

$$\Gamma_{g_1,1}^{n_1-1}\times\Gamma_{g_2,1}^{n_2-1}\to\Gamma_g^n.$$

By a well known exact sequence

$$1\to\mathbb{Z}\to\Gamma_{g_i,1}^{n_i-1}\to\Gamma_{g_i}^{n_i}\to 1,$$

this induces a homomorphism

$$p:\Gamma_{g_1}^{n_1}[3]\times\Gamma_{g_2}^{n_2}[3]\to\Gamma_g^n,$$

which preserves the filtration (Nakamura [14]).

In his paper [14], Nakamura has shown that *if n is greater than* 0, this homomorphism preserves the filtration strictly, i.e.,

$$p^{-1}\bigl(\Gamma_g^n[m]\bigr)=\Gamma_{g_1}^{n_1}[m]\times\Gamma_{g_2}^{n_2}[m]\quad(m\geqslant 3)$$

holds. Our first result is to show that *this holds also in the case that $n=0$* (Theorem A).

The motivation of giving this supplementary result is explained as follows. The filtration $\{\Gamma_g^n[m]\}_{m=1}^{\infty}$ has been investigated from the viewpoint of both topological and arithmetic interests. (Cf., e.g., Morita [12, 13] and [14], respectively.) It is known that the graded quotient module $\mathrm{gr}^m(\Gamma_g^n)(=\Gamma_g^n[m]/\Gamma_g^n[m+1])$ is a free $\mathbb{Z}$-module of finite rank on which $\mathrm{Sp}(2g,\mathbb{Z})$ naturally acts. Therefore, to determine the $\mathrm{Sp}(2g,\mathbb{Z})$-module structure of $\mathrm{gr}^m(\Gamma_g^n)$ seems to be a fundamental problem. (The $\mathrm{Sp}(2g,\mathbb{Q})$-module structure of $\mathrm{gr}^m(\Gamma_g^n)\otimes\mathbb{Q}$ has been determined for small m (Asada–Nakamura and Morita, cf. [3]).) But even the rank of $\mathrm{gr}^m(\Gamma_g^n)$ has not yet been determined (except in the case of $g=0,1$, see Section 5).

A few years ago, using the "cycle twists" due to Matsumoto [11], Oda [16] constructed a subgroup P of $\Gamma_g(=\Gamma_g^0)$, the mapping class group of a compact surface. Here P is isomorphic to the Artin pure braid group with g-strings. He showed that the inclusion $P\to\Gamma_g$ induces injective homomorphisms

$$\mathrm{gr}^m(P)\to\mathrm{gr}^{m-1}(\Gamma_g)\quad(m=2,3,\ldots),$$

$\mathrm{gr}^m(P)$ being the graded quotient module associated with the lower central filtration of P. As a consequence, he obtained a lower bound of the rank of $\mathrm{gr}^m(\Gamma_g)$.

Now, the method of pasting two surfaces gives us another way of obtaining a lower bound for the rank of $\mathrm{gr}^m(\Gamma_g)$. Namely, the case that $(g_2,n_2)=(1,1)$ gives an injective homomorphism

$$\mathrm{gr}^m(\Gamma_{g-1}^{n+1})\to\mathrm{gr}^m(\Gamma_g^n).$$

Composing these, we have an injective homomorphism

$$\mathrm{gr}^m(\Gamma_1^{g-1})\to\mathrm{gr}^m(\Gamma_g).$$

On the other hand, the rank of $\mathrm{gr}^m(\Gamma_1^n)$ can be calculated comparatively easily. Thus, we obtain the following:

Theorem B (Section 5). *For $m \geqslant 1$ and $g \geqslant 3$, we have*

$$\operatorname{rank}\mathrm{gr}^m(\Gamma_g) \geqslant \frac{1}{m}\sum_{r=1}^{g-2}\sum_{d|m}\mu(d)\bigl\{(1+\sqrt{r})^{m/d}+(1-\sqrt{r})^{m/d}\bigr\}.$$

The right hand side of the above inequality is a polynomial of g with rational coefficients whose degree is $[m/2]+1$.

In general, constructing linearly independent elements of $\mathrm{gr}^m(\Gamma_g^n)$ and obtaining a lower bound for its rank seems to be more difficult in the compact case (i.e., in the case of $n=0$) than in the non-compact case. The reason is that, if $n=0$, $\pi_1(\Sigma_g^n)$ is not a free group and has one relation. As is well known, this one relator group is an amalgam of two free groups. The proof of Theorem A will be done by using the amalgam decomposition of, not the fundamental group itself, but its graded Lie algebra associated with the weight filtrations.

The organization of the present paper is as follows. In Section 1, we review the definition and basic properties of the filtration of mapping class groups. In Section 2, the pasting homomorphism is defined and the proof of Theorem A will be reduced to two properties of the graded Lie algebras associated with the weight filtration of fundamental groups. In Section 3, one of the properties will be verified. The proof of Theorem A will be completed in Section 4. In Section 5, the rank of $\mathrm{gr}^m(\Gamma_g^n)$ for $g=0,1$ will be calculated and the proof of Theorem B will be given. In the case that $g=0,1$, the central filtration $\{\Gamma_g^n[m]\}_{m\geqslant 1}$ of $\Gamma_g^n[1]$ (essentially) coincides with the lower central filtration of $\Gamma_g^n[1]$. This (more or less well known) fact will also be verified.

1. Filtrations of mapping class groups

In this section, we recall the definition of the filtration of mapping class groups and summarize some basic properties (cf. Asada [2, 3], Nakamura and Tsunogai [15]).

(1.1) Let $\pi_{g,n}$ be the group defined by the following generators and a defining relation:

$$\begin{aligned}\text{generators:}\quad & x_1,\ldots,x_{2g},z_1,\ldots,z_n,\\ \text{relation:}\quad & [x_1,x_{g+1}]\cdots[x_g,x_{2g}]z_1\cdots z_n=1.\end{aligned}$$

Here the bracket [,] denotes the commutator: $[a,b]=aba^{-1}b^{-1}$. We assume $g\geqslant 1$, $n\geqslant 0$ and $2-2g-n<0$.

The weight filtration (due to Oda and Kaneko [8]) $\{\pi_{g,n}(m)\}_{m=1}^{\infty}$ of $\pi_{g,n}$ is defined as follows:

$$\begin{aligned}&\pi_{g,n}(1)=\pi_{g,n},\\ &\pi_{g,n}(2)=[\pi_{g,n},\pi_{g,n}]\langle z_1,\ldots,z_n\rangle,\\ &\pi_{g,n}(m)=\bigl\langle[\pi_{g,n}(m'),\pi_{g,n}(m'')]\mid m'+m''=m\bigr\rangle\quad(m\geqslant 3).\end{aligned}$$

Then, $\{\pi_{g,n}(m)\}_{m=1}^{\infty}$ is a decreasing sequence of normal subgroups of $\pi_{g,n}$ and

$$\bigcap_{m\geqslant 1}\pi_{g,n}(m)=\{1\}.$$

Moreover, it is a central filtration, i.e.,

$$\bigl[\pi_{g,n}(m),\pi_{g,n}(m')\bigr]\subset\pi_{g,n}(m+m')$$

holds for all $m, m'\geqslant 1$. Hence, the graded module

$$\operatorname{gr}(\pi_{g,n})=\bigoplus_{m=1}^{\infty}\operatorname{gr}^m(\pi_{g,n})\quad\bigl(\operatorname{gr}^m(\pi_{g,n})=\pi_{g,n}(m)/\pi_{g,n}(m+1)\bigr)$$

has the structure of a graded Lie algebra over $\mathbb{Z}$, the Lie bracket being naturally induced from the commutator (cf. Bourbaki [6]).

The fundamental results about this Lie algebra we need are summarized as follows.

Theorem WL (Witt [18], Labute [9]).

(i) *The generators and a defining relation of the Lie algebra* $\operatorname{gr}(\pi_{g,n})$ *are given as follows*:

$$\textit{generators}:\quad X_1,\dots,X_{2g},Z_1,\dots,Z_n,$$
$$\textit{relation}:\quad \sum_{i=1}^{g}[X_i,X_{g+i}]+\sum_{j=1}^{n}Z_j=0,$$

where $X_i=x_i \bmod \pi_{g,n}(2)$ $(i=1,\dots,2g)$ *and* $Z_j=z_j \bmod \pi_{g,n}(3)$ $(j=1,\dots,n)$.

(ii) *For each* m, $\operatorname{gr}^m(\pi_{g,n})$ *is a finitely generated free* $\mathbb{Z}$*-module whose rank is given by*

$$\operatorname{rank}\operatorname{gr}^m(\pi_{g,n})=\frac{1}{m}\sum_{d\mid m}\mu(d)\bigl(\alpha^{m/d}+\beta^{m/d}\bigr),$$

where $\alpha=g+\sqrt{g^2+n-1}$, $\beta=g-\sqrt{g^2+n-1}$ *and* μ *denotes the Möbius function.*

(iii) *The universal enveloping algebra of* $\operatorname{gr}(\pi_{g,n})$ *is the associative algebra over* $\mathbb{Z}$ *having the following generators and a defining relation*:

$$\textit{generators}:\quad X_1,\dots,X_{2g},Z_1,\dots,Z_n,$$
$$\textit{relation}:\quad \sum_{i=1}^{g}(X_iX_{g+i}-X_{g+i}X_i)+\sum_{j=1}^{n}Z_j=0.$$

(1.2) We define the group of "braid-like" automorphisms of $\pi_{g,n}$ by

$$\operatorname{Aut}_{\{z_i\}_i}(\pi_{g,n})=\bigl\{\sigma\in\operatorname{Aut}(\pi_{g,n})\mid\sigma(z_j)\sim z_j\ (j=1,\dots,n)\bigr\},$$

where $\sim$ denotes conjugacy in $\pi_{g,n}$.

Since each element of $\operatorname{Aut}_{\{z_i\}_i}(\pi_{g,n})$ stabilizes $\pi_{g,n}(2)$, $\operatorname{Aut}_{\{z_i\}_i}(\pi_{g,n})$ acts on $\pi_{g,n}/\pi_{g,n}(2)\simeq\mathbb{Z}^{2g}$. From this we get a representation

$$\begin{aligned} \rho : \mathrm{Aut}_{\{z_i\}_i}(\pi_{g,n}) &\to \mathrm{GL}(2g, \mathbb{Z}), \\ \sigma &\mapsto (\rho_{ij}), \end{aligned}$$

where

$$\sigma(x_i) \equiv \prod_{k=1}^{2g} x_k^{\rho_{ki}} \bmod \pi_{g,n}(2).$$

By a classical result of Nielsen, the image of this representation coincides with

$$\mathrm{GSp}(2g, \mathbb{Z}) = \left\{ A \in \mathrm{GL}(2g, \mathbb{Z}) \mid {}^t A J_g A = \chi(A) J_g,\ \chi(A) = \pm 1 \right\},$$

where

$$J_g = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}.$$

We define the group $\widetilde{\Gamma}_g^n$ by

$$\widetilde{\Gamma}_g^n = \left\{ \sigma \in \mathrm{Aut}_{\{z_i\}_i}(\pi_{g,n}) \mid \chi(\rho(\sigma)) = 1 \right\}.$$

The quotient group of $\widetilde{\Gamma}_g^n$ by the inner automorphism group $\mathrm{Int}(\pi_{g,n})$ is denoted by Γ_g^n:

$$\Gamma_g^n = \widetilde{\Gamma}_g^n / \mathrm{Int}(\pi_{g,n}).$$

In the case that $n \geqslant 1$, we also define the group $\Gamma_{g,1}^{n-1}$ by

$$\Gamma_{g,1}^{n-1} = \left\{ \sigma \in \widetilde{\Gamma}_g^n \mid \sigma(z_n) = z_n \right\}.$$

Since the centralizer of z_n in $\pi_{g,n}$ is the infinite cyclic group $\mathbb{Z}$ generated by z_n, the canonical homomorphism $\Gamma_{g,1}^{n-1} \to \Gamma_g^n$ induces an exact sequence

$$1 \to \mathbb{Z} \to \Gamma_{g,1}^{n-1} \to \Gamma_g^n \to 1. \tag{1.2.1}$$

As is well known, the group Γ_g^n (respectively $\Gamma_{g,1}^{n-1}$) is isomorphic to the mapping class group of a genus g surface with n-points preserved (respectively with $(n-1)$-points preserved and 1 boundary component fixed) (cf., e.g., Birman [5]).

(1.3) For each non-negative integer m, set

$$\widetilde{\Gamma}_g^n[m] = \left\{ \sigma \in \widetilde{\Gamma}_g^n \;\middle|\; \begin{array}{l} \sigma(x_i)x_i^{-1} \in \pi_{g,n}(m+1)\ (i = 1, \ldots, 2g) \\ \sigma(z_j) \overset{m}{\sim} z_j\ (j = 1, \ldots, n) \end{array} \right\} \quad (m \geqslant 0).$$

Here, $\overset{m}{\sim}$ denotes conjugacy by an element of $\pi_{g,n}(m)$. Then, $\{\widetilde{\Gamma}_g^n[m]\}_{m=0}^{\infty}$ gives a decreasing sequence of normal subgroups of $\widetilde{\Gamma}_g^n$. Moreover, $\{\widetilde{\Gamma}_g^n[m]\}_{m=1}^{\infty}$ is a central filtration of the group $\widetilde{\Gamma}_g^n[1]$.

The filtration $\{\widetilde{\Gamma}_g^n[m]\}_{m=0}^{\infty}$ of $\widetilde{\Gamma}_g^n$ naturally induces a filtration of Γ_g^n and $\Gamma_{g,1}^{n-1}$; namely we put

$$\begin{aligned} &\Gamma_g^n[m] = \widetilde{\Gamma}_g^n[m]\mathrm{Int}(\pi_{g,n})/\mathrm{Int}(\pi_{g,n}) \quad (m \geqslant 0), \\ &\Gamma_{g,1}^{n-1}[m] = \Gamma_{g,1}^{n-1} \cap \widetilde{\Gamma}_g^n[m] \quad (m \geqslant 0). \end{aligned}$$

Since $\mathrm{Int}(z_n) \in \Gamma_g^n[2] \setminus \Gamma_g^n[3]$, from (1.2.1) we have the following:

Lemma 1. *For $m \geqslant 3$, the canonical homomorphism*

$$\Gamma_{g,1}^{n-1}[m] \to \Gamma_g^n[m]$$

is an isomorphism.

(1.4) To describe the module $\mathrm{gr}^m(\Gamma_g^n)$ $(m \geqslant 1)$, let us recall the coordinate module and its submodules (cf. [15]).

The coordinate module $C^m(2g,n)$ $(m \geqslant 1)$ is defined by

$$C^m(2g,n) = \begin{cases} (\mathrm{gr}^{m+1}(\pi_{g,n}))^{\oplus 2g} \oplus (\mathrm{gr}^m(\pi_{g,n}))^{\oplus n} & (m \neq 2), \\ (\mathrm{gr}^{m+1}(\pi_{g,n}))^{\oplus 2g} \oplus \bigoplus_{j=1}^n (\mathrm{gr}^m(\pi_{g,n})/\mathbb{Z}\bar{z}_j) & (m = 2). \end{cases}$$

Here, $^-$ denotes the image in the quotient. We define the mapping

$$c_m(2g,n) : \widetilde{\Gamma}_g^n[m] \to C^m(2g,n)$$

as follows. For $\sigma \in \widetilde{\Gamma}_g^n[m]$, put $s_i(\sigma) = \sigma(x_i)x_i^{-1}$ $(1 \leqslant i \leqslant 2g)$, and let t_j be an element of $\pi_{g,n}(m)$ such that $\sigma(z_j) = t_j z_j t_j^{-1}$ $(1 \leqslant j \leqslant n)$. Since the centralizer of z_j in $\pi_{g,n}$ is the infinite cyclic subgroup generated by z_j, t_j is uniquely determined if $m \neq 2$, and $\bar{t}_j \bmod \mathbb{Z}\bar{z}_j$ is uniquely determined if $m = 2$. We define

$$c_m(2g,n)(\sigma) = \left(\overline{s_i(\sigma)}\right)_{1 \leqslant i \leqslant 2g} \times (\bar{t}_j)_{1 \leqslant j \leqslant n}.$$

Then, $c_m(2g,n)$ induces an injective homomorphism

$$\tilde{\iota}_m(2g,n) : \mathrm{gr}^m(\widetilde{\Gamma}_g^n) \to C^m(2g,n).$$

Let us consider the linear homomorphism

$$\begin{aligned} \mathrm{gr}^m(\pi_{g,n}) &\to C^m(2g,n) \\ \bar{t} &\mapsto (\overline{[t,x_i]})_i \times (\bar{t},\ldots,\bar{t}) \end{aligned}$$

and denote its image by $I^m(2g,n)$. The module $I^m(2g,n)$ is contained in the image of $\tilde{\iota}_m(2g,n)$. The quotient module $C^m(2g,n)/I^m(2g,n)$ is denoted by $\overline{C}^m(2g,n)$.

Proposition 1.
(i) *As a $\mathbb{Z}$-module, $\overline{C}^m(2g,n)$ is free.*
(ii) *The homomorphism $\tilde{\iota}_m(2g,n)$ induces an injective homomorphism*

$$\iota_m(2g,n) : \mathrm{gr}^m(\Gamma_g^n) \to \overline{C}^m(2g,n).$$

For the proof, cf., e.g., [2].

In the case that $n \geqslant 1$, we define a submodule $C^m(2g,n)^*$ of $C^m(2g,n)$ by

$$C^m(2g,n)^* = \left\{(S_i) \times (T_j) \in C^m(2g,n) \mid T_n = 0\right\},$$

and denote the restriction of $\tilde{\iota}_m(2g,n)$ to $\mathrm{gr}^m(\Gamma_{g,1}^{n-1}) (\subset \mathrm{gr}^m(\widetilde{\Gamma}_g^n))$ by $\iota_m^*(2g,n)$.

Thus, three kind of graded modules are related by the following diagram:

$$\begin{array}{ccccc}
\mathrm{gr}^m(\Gamma_{g,1}^{n-1}) & \longrightarrow & \mathrm{gr}^m(\widetilde{\Gamma}_g^n) & \longrightarrow & \mathrm{gr}^m(\Gamma_g^n) \\
\downarrow{\scriptstyle \iota_m^*} & & \downarrow{\scriptstyle \tilde{\iota}_m} & & \downarrow{\scriptstyle \iota_m} \\
C^m(2g,n)^* & \longrightarrow & C^m(2g,n) & \longrightarrow & \overline{C}^m(2g,n)
\end{array}$$

The following lemma will be used in the proof of Theorem A.

Lemma 2. *Assume that $m \geqslant 3$. If the image of $\iota_m^*(2g,1)$ contains an element of the form*

$$\big([U, X_1], \ldots, [U, X_{2g}], 0\big)$$

with $U \in \mathrm{gr}^m(\pi_{g,1})$, then $U = 0$.

Proof. Let σ be an element of $\Gamma_{g,1}[m]$ so that

$$\sigma(x_i) = s_i x_i \quad (1 \leqslant i \leqslant 2g),$$
$$\sigma(z_1) = z_1$$

with $s_i \in \pi_{g,1}(m+1)$. Assume that there exists an element u of $\pi_{g,1}(m)$ satisfying

$$s_i \equiv [u, x_i] \bmod \pi_{g,1}(m+2) \quad (1 \leqslant i \leqslant 2g).$$

We have to show that $u \in \pi_{g,1}(m+1)$. Put $\tau = \mathrm{Int}(u)^{-1}\sigma$ ($\in \mathrm{Aut}(\pi_{g,1})$). Then, $\tau(x_i)x_i^{-1} \in \pi_{g,1}(m+2)$ $(1 \leqslant i \leqslant 2g)$ and $\tau(z_1) = u^{-1}z_1u$. As $\pi_{g,1}$ is generated by x_i $(1 \leqslant i \leqslant 2g)$, this shows that τ acts trivially on $\pi_{g,1}/\pi_{g,1}(m+2)$. Hence τ acts trivially on $\pi_{g,1}(2)/\pi_{g,1}(m+3)$. Thus $\tau(z_1)z_1^{-1} = [u^{-1}, z_1] \in \pi_{g,1}(m+3)$ as $z_1 \in \pi_{g,1}(2)$. Therefore, if we put $U = u \bmod \pi_{g,1}(m+1)$, we have

$$\left[U, \sum_{i=1}^{g}[X_i, X_{g+i}]\right] = 0$$

in the free Lie algebra $\mathrm{gr}(\pi_{g,1})$. Since U and $\sum_{i=1}^{g}[X_i, X_{g+i}]$ are homogeneous of degree m and 2, respectively, they are linearly dependent (cf. Magnus et al. [10, Th. 5.10]). As $m \geqslant 3$, we have $U = 0$, i.e., $u \in \pi_{g,1}(m+1)$. □

2. Pasting homomorphisms

(2.1) Let π_{g_1,n_1} $(n_1 \geqslant 1)$ and π_{g_2,n_2} $(n_2 \geqslant 1)$ be the groups defined by the following generators and relations:

$$\pi_{g_1,n_1}\begin{cases} \text{generators: } x_1, \ldots, x_{2g_1}, z_1, \ldots, z_{n_1} \\ \quad \text{relation: } [x_1, x_{g_1+1}]\cdots[x_{g_1}, x_{2g_1}]z_1\cdots z_{n_1} = 1, \end{cases}$$

$$\pi_{g_2,n_2}\begin{cases} \text{generators: } y_1, \ldots, y_{2g_2}, w_1, \ldots, w_{n_2} \\ \quad \text{relation: } [y_1, y_{g_2+1}]\cdots[y_{g_2}, y_{2g_2}]w_1\cdots w_{n_2} = 1. \end{cases}$$

As π_{g_1,n_1} is the free group on $x_1,\dots,x_{2g_1},z_1,\dots,z_{n_1-1}$, it is easy to see that the subgroup $\langle z_{n_1}\rangle$ generated by z_{n_1} is an infinite cyclic group. Similarly, the subgroup $\langle w_{n_2}\rangle$ of π_{g_2,n_2} is also an infinite cyclic group. Fix isomorphisms as follows:

$$\begin{aligned}\langle z_{n_1}\rangle &\simeq \mathbb{Z} \simeq \langle w_{n_2}\rangle,\\ z_{n_1} &\leftrightarrow\ 1\ \leftrightarrow w_{n_2}^{-1}.\end{aligned}$$

The amalgamated product of π_{g_1,n_1} and π_{g_2,n_2} with respect to $\mathbb{Z}$ is isomorphic to $\pi_{g,n}$ $(g=g_1+g_2,\ n=n_1+n_2-2)$:

$$\pi_{g,n}=\pi_{g_1,n_1}*_{\mathbb{Z}}\pi_{g_2,n_2}.$$

Let σ_1 and σ_2 be elements of $\Gamma^{n_1-1}_{g_1,1}$ and $\Gamma^{n_2-1}_{g_2,1}$, respectively. By definition, σ_1 (respectively σ_2) is the identity on the subgroup $\langle z_{n_1}\rangle$ (respectively $\langle w_{n_2}\rangle$). Hence the pair (σ_1,σ_2) of automorphisms determines an automorphism of $\pi_{g,n}$. We denote this by $\sigma_1*\sigma_2$:

$$\sigma_1*\sigma_2:\pi_{g,n}\to\pi_{g,n}.$$

Obviously, $\sigma_1*\sigma_2$ belongs to $\widetilde{\Gamma}^n_g$. Then, we have a homomorphism

$$\begin{aligned}\tilde{p}:\Gamma^{n_1-1}_{g_1,1}\times\Gamma^{n_2-1}_{g_2,1}\ &\to\ \widetilde{\Gamma}^n_g\\ (\sigma_1,\sigma_2)\ &\mapsto\ \sigma_1*\sigma_2.\end{aligned}$$

It is easy to see that this homomorphism preserves the filtration, i.e., if $\sigma_i\in\Gamma^{n_i-1}_{g_i,1}[m]$ $(i=1,2)$, then $\sigma_1*\sigma_2\in\widetilde{\Gamma}^n_g[m]$. Composing this with the canonical homomorphism $\widetilde{\Gamma}^n_g[3]\to\Gamma^n_g[3]$, we have a homomorphism

$$\Gamma^{n_1-1}_{g_1,1}[3]\times\Gamma^{n_2-1}_{g_2,1}[3]\to\Gamma^n_g[3],$$

which preserves the filtration. By Lemma 1, this may be regarded as a homomorphism

$$\Gamma^{n_1}_{g_1}[3]\times\Gamma^{n_2}_{g_2}[3]\to\Gamma^n_g[3].$$

Both homomorphisms will be denoted by p. Then, our main result is the following:

Theorem A. *The homomorphism p preserves the filtration strictly, i.e.,*

$$p^{-1}\big(\Gamma^n_g[m]\big)=\Gamma^{n_1-1}_{g_1,1}[m]\times\Gamma^{n_2-1}_{g_2,1}[m]\quad(m\geqslant 3).$$

(2.2) By using the coordinate modules, the above theorem can be interpreted as giving some properties of the graded Lie algebras $\mathrm{gr}(\pi_{g_i,n_i})$ $(i=1,2)$ and $\mathrm{gr}(\pi_{g,n})$ as follows.

Let

$$f_i:\pi_{g_i,n_i}\to\pi_{g,n}=\pi_{g_1,n_1}*_{\mathbb{Z}}\pi_{g_2,n_2}\quad(i=1,2)$$

be the canonical homomorphisms. As is well known, these are injective. It is easy to see that these homomorphisms preserve the weight filtration. Hence they induce homomorphisms of the associated graded Lie algebras

$$\mathrm{gr}(f_i):\mathrm{gr}(\pi_{g_i,n_i})\to\mathrm{gr}(\pi_{g,n})\quad(i=1,2).$$

Let us consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{gr}^m(\Gamma_{g_1,1}^{n_1-1}) \oplus \mathrm{gr}^m(\Gamma_{g_2,1}^{n_2-1}) \arrow[r, "\mathrm{gr}^m(\tilde{p})"] \arrow[d, "\iota^*"'] & \mathrm{gr}^m(\widetilde{\Gamma}_g^n) \arrow[r] \arrow[d, "\tilde{\iota}_m"'] & \mathrm{gr}^m(\Gamma_g^n) \arrow[d, "\iota_m"'] \\
C^m(2g_1,n_1)^* \oplus C^m(2g_2,n_2)^* \arrow[r, "F"] & C^m(2g,n) \arrow[r] & \overline{C}^m(2g,n)
\end{tikzcd}.$$

Here, ι^* is the sum of $\iota_m^*(g_1,n_1)$ and $\iota_m^*(g_2,n_2)$ and F is the homomorphism naturally induced from $\mathrm{gr}(f_1)$ and $\mathrm{gr}(f_2)$. The above theorem asserts that

$$\mathrm{gr}^m(p): \mathrm{gr}^m(\Gamma_{g_1,1}^{n_1-1}) \oplus \mathrm{gr}^m(\Gamma_{g_2,1}^{n_2-1}) \to \mathrm{gr}^m(\Gamma_g^n)$$

is injective. Therefore, *the proof of Theorem A reduces to showing that*

(I) *F is injective and*

(II) $\mathrm{Im}(F \circ \iota^*) \cap I^m(2g,n) = \{0\}$ $(m \geqslant 3)$.

3. Amalgamated product of rings

(3.1) To prove the statements (I) and (II) in (2.2), we need the structure theorem of amalgams of rings, which we briefly recall.

Let A, B_1, B_2 be rings and $A \to B_i$ be an injective homomorphism for $i = 1, 2$. Let B be the amalgamated product of B_1 and B_2 with respect to A:

$$B = B_1 *_A B_2.$$

Here, we make the following assumption:

(A) *There exists a sub-A-bimodule B_i' of B_i such that $B_i = A \oplus B_i'$* $(i = 1, 2)$.

Then, we have the following:

Structure Theorem.

$$B = A \oplus \Bigl(\bigoplus B_{\boldsymbol{i}}'\Bigr).$$

Here, $B_{\boldsymbol{i}}'$ is a sub-A-bimodule of B and

$$B_{\boldsymbol{i}}' \simeq B_{i_1}' \otimes_A B_{i_2}' \otimes_A \cdots \otimes_A B_{i_n}' \quad (\boldsymbol{i} = (i_1, i_2, \ldots, i_n),\ i_m \neq i_{m+1}).$$

(Cf. Serre [17, 1.2 Exercise].)

Corollary. *The canonical homomorphism $B_i \to B$ is injective.*

(3.2) Let B_1 (respectively B_2) be the free associative algebra on $X_1, \ldots, X_{2g_1}, Z_1, \ldots, Z_{n_1-1}$ (respectively $Y_1, \ldots, Y_{2g_2}, W_1, \ldots, W_{n_2-1}$) over $\mathbb{Z}$.

Put

$$Z = \sum_{i=1}^{g_1} (X_i X_{g_1+i} - X_{g_1+i} X_i) + \sum_{j=1}^{n_1-1} Z_j,$$

$$W=\sum_{i=1}^{g_2}(Y_iY_{g_2+i}-Y_{g_2+i}Y_i)+\sum_{j=1}^{n_2-1}W_j,$$

and let $A=\mathbb{Z}[t]$ be the polynomial ring of one variable over $\mathbb{Z}$. Then, it is easy to see that the subalgebra $\langle Z\rangle$ of B_1 (respectively $\langle W\rangle$ of B_2) generated by Z (respectively by W) is isomorphic to A. Fix isomorphisms as follows:

$$\begin{aligned}\langle Z\rangle &\simeq A \simeq \langle W\rangle\\ Z &\leftrightarrow t \leftrightarrow -W\end{aligned}$$

Let B be the amalgamated product of B_1 and B_2 with respect to A. Thus B is an associative algebra over $\mathbb{Z}$ and has the following generators and a relation:

generators: $X_1,\dots,X_{2g_1},Z_1,\dots,Z_{n_1-1},Y_1,\dots,Y_{2g_2},W_1,\dots,W_{n_2-1}$,

$$\text{relation: }\sum_{i=1}^{g_1}(X_iX_{g_1+i}-X_{g_1+i}X_i)+\sum_{j=1}^{n_1-1}Z_j+\sum_{i=1}^{g_2}(Y_iY_{g_2+i}-Y_{g_2+i}Y_i)+\sum_{j=1}^{n_2-1}W_j=0.$$

To apply the Structure Theorem, we verify the following:

Lemma 3. *The assumption* (A) *is satisfied for these* B_i.

Proof. We shall prove the lemma for B_1. For simplicity, we write g for g_1. We regard B_1 as a graded algebra over $\mathbb{Z}$, where the degree of X_i (respectively Z_j) is 1 (respectively 2).

Let $B_1^{(n)}$ $(n\geqslant 1)$ be the set of elements of homogeneous of degree n and $v_1^{(n)},v_2^{(n)},\dots,v_{r_n}^{(n)}$ be all monomials of degree n. Then, $B_1^{(n)}$ is a free $\mathbb{Z}$-module with a basis $v_1^{(n)},v_2^{(n)},\dots,v_{r_n}^{(n)}$.

Let X_l be given. We may (and will) assume that $v_1^{(2m)}=(X_lX_{g+l})^m$ $(m\geqslant 1)$. Since Z^m is a linear combination of distinct monomials with coefficients ± 1,

$$Z^m,v_2^{(2m)},\dots,v_{r_{2m}}^{(2m)}$$

is also a basis of $B_1^{(2m)}$.

Put

$$\begin{aligned}C&=\bigoplus_{m=1}^{\infty}B_1^{(2m-1)},\\ D&=\bigoplus_{m=1}^{\infty}\bigl(\mathbb{Z}v_2^{(2m)}\oplus\cdots\oplus\mathbb{Z}v_{r_{2m}}^{(2m)}\bigr).\end{aligned}$$

Claim 1. $ZC\subset C$, $CZ\subset C$.

This is obvious, as Z is homogeneous of degree 2.

Claim 2. $ZD \subset D$, $DZ \subset D$.

As $Zv_k^{(2m)} \in B_1^{(2m+2)}$ $(2 \leqslant k \leqslant r_{2m})$, we have

$$Zv_k^{(2m)} = c_1 Z^{m+1} + c_2 v_2^{(2m+2)} + \cdots + c_{r_{2m+2}} v_{2m+2}^{(2m+2)} \quad (c_i \in \mathbb{Z}).$$

Comparing the coefficients of $(X_l X_{g+l})^{m+1}$ in both sides, we have $c_1 = 0$. Thus, $Zv_k^{(2m)} \in D$. Therefore, $ZD \subset D$. Similarly, $DZ \subset D$.

Put $B_1' = C \oplus D$. Then, by Claims 1 and 2, B_1' is a sub-A-bimodule and $B_1 = A \oplus B_1'$. $\square$

Remark. Assume that $g_1 \geqslant 2$ and X_k $(1 \leqslant k \leqslant g_1)$ are given. Choose X_l $(1 \leqslant l \leqslant g_1)$ different from X_k. Then it can be checked easily that the sub-A-bimodule B_1' constructed in the proof of Lemma 3 satisfies $X_k B_1' \subset B_1'$, $B_1' X_k \subset B_1'$. This fact will be used later.

(3.3) As the first application of the Structure Theorem, we have the following:

Lemma 4. *The homomorphisms* $\mathrm{gr}(f_i)$ $(i = 1, 2)$ *are injective.*

Proof. By Theorem WL, the algebras B_1, B_2 and B are the universal enveloping algebras of the Lie algebras $\mathrm{gr}(\pi_{g_1,n_1})$, $\mathrm{gr}(\pi_{g_2,n_2})$ and $\mathrm{gr}(\pi_{g,n})$, respectively. Moreover, these Lie algebras are free as $\mathbb{Z}$-modules. Therefore, by the Poincaré–Birkhoff–Witt theorem, the lemma follows from the Corollary of the Structure Theorem. $\square$

Corollary. *The homomorphism F in* (2.2) *is injective.*

4. Proof of Theorem A

(4.1) In this section, we shall prove Theorem A. As mentioned in the introduction, it is proved in [14] in the case that either $n_1 > 1$ or $n_2 > 1$. Therefore, we shall restrict ourselves to the case that $n_1 = n_2 = 1$.

It remains to verify the statement (II) in (2.2).

Let L be the Lie algebra over $\mathbb{Z}$ defined by the following generators and a relation:

generators: $X_1, \ldots, X_{2g_1}, Y_1, \ldots, Y_{2g_2}$,

$$\text{relation: } \sum_{i=1}^{g_1} [X_i, X_{g_1+i}] + \sum_{j=1}^{g_2} [Y_j, Y_{g_2+j}] = 0.$$

Let L_1 (respectively L_2) be the Lie subalgebra of L generated by $X_1, \ldots, X_{2g_1}$ (respectively $Y_1, \ldots, Y_{2g_2}$). Thus, by Lemma 4, $L_i = \mathrm{gr}(\pi_{g_i,1})$ $(i = 1, 2)$ and $L = \mathrm{gr}(\pi_{g,0})$ with $g = g_1 + g_2$.

Then the module $I^m(2g, 0)$ consists of all elements of the form

$$\big([U, X_1], \ldots, [U, X_{2g_1}], [U, Y_1], \ldots, [U, Y_{2g_2}]\big) \tag{4.1.1}$$

with $U \in \mathrm{gr}^m(\pi_{g,0})$.

(4.2) First we treat the case that one of g_i, say g_2, is 1. As is well known, the group $\Gamma_{1,1}[1] = \{1\}$, hence $\mathrm{gr}^m(\Gamma_{1,1}) = \{0\}$. Therefore, if the image of $F \circ \iota^*$ contains an element of the form (4.1.1), we have

$$[U, Y_1] = [U, Y_2] = 0.$$

Thus, it suffices to show the following:

Lemma 5. *Let U be an element of $\mathrm{gr}(\pi_{g,0})$ satisfying $[U, Y_1] = [U, Y_2] = 0$. Then, $U = 0$.*

For the proof, see [3, Lemma 3.13].

(4.3) We shall prove Theorem A in the case that both of the g_i are greater than 1. Instead of Lemma 5 in the case that $g_2 = 1$, we need the following lemma whose proof will be given later.

Lemma 6. *Assume that $g_1 \geqslant 2$ and $g_2 \geqslant 2$. Let X_{k_1} and Y_{k_2} be given and let U be an element of L satisfying $[U, X_{k_1}] \in L_1$ and $[U, Y_{k_2}] \in L_2$. Then, $U \in \mathbb{Z} \cdot Z$ (hence $U \in L_1 \cap L_2$), where*

$$Z = \sum_{i=1}^{g_1} [X_i, X_{g_1+i}].$$

Assume that the image of $F \circ \iota^*$ contains an element of the form (4.1.1). Then we have

$$[U, X_i] \in L_1 \quad (1 \leqslant i \leqslant 2g_1),$$
$$[U, Y_j] \in L_2 \quad (1 \leqslant j \leqslant 2g_2).$$

By Lemma 6, U belongs to $L_1 \cap L_2$. Thus the image of $\iota_m^*(2g_1, 1)$ contains the element

$$\bigl([U, X_1], \ldots, [U, X_{2g_1}], 0\bigr),$$

where U belongs to L_1. Then $U = 0$ by Lemma 2 and the proof is complete.

Proof of Lemma 6. Let B_1, B_2 and B be the algebras defined in (3.2). Then, by Theorem WL, these are the universal enveloping algebras of L_1, L_2 and L, respectively. We shall consider that these Lie algebras and algebras are (via canonical homomorphisms) all contained in the algebra B. Let B_1' be the sub-A-bimodule of B_1 constructed in the proof of Lemma 3, where we choose X_l different from X_{k_1}. (This is possible as $g_1 \geqslant 2$.) Let B_2' denote a sub-A-bimodule of B_2 constructed in a similar way. Then, as is remarked after Lemma 3, we have $X_{k_1} B_1' \subset B_1'$, $B_1' X_{k_1} \subset B_1'$ and $Y_{k_2} B_2' \subset B_2'$, $B_2' Y_{k_2} \subset B_2'$.

By the Structure Theorem, U can be expressed as

$$U = S + \sum_{\boldsymbol{i}} T_{\boldsymbol{i}}$$

with $S \in A$ and $T_{\boldsymbol{i}} \in B_{\boldsymbol{i}}'$ ($\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$, $i_m \neq i_{m+1}$).

Claim 1. *$T_{\boldsymbol{i}} = 0$ if $n \geqslant 2$.*

We shall prove this in the case that $i_n = 2$. The case $i_n = 1$ can be proved in the same way and will be omitted.

Case 1. $i_1 = 1$. Put $\boldsymbol{i}_0 = (i_2, \ldots, i_n)$. Then, the $\boldsymbol{i}$-component of UX_{k_1} is 0 and that of $X_{k_1}U$ is

$$X_{k_1}T_{\boldsymbol{i}_0} + X_{k_1}T_{\boldsymbol{i}} = X_{k_1}(T_{\boldsymbol{i}_0} + T_{\boldsymbol{i}}).$$

As $UX_{k_1} - X_{k_1}U$ belongs to $B_1 = A \oplus B_1'$, we have $X_{k_1}(T_{\boldsymbol{i}_0} + T_{\boldsymbol{i}}) = 0$. Hence, $T_{\boldsymbol{i}_0} + T_{\boldsymbol{i}} = 0$. Therefore, $T_{\boldsymbol{i}_0} = T_{\boldsymbol{i}} = 0$.

Case 2. $i_1 = 2$. Put $\boldsymbol{i}' = (i_1, \ldots, i_n, 1)$. Then the $\boldsymbol{i}'$-component of $X_{k_1}U$ is 0 and that of UX_{k_1} is

$$T_{\boldsymbol{i}}X_{k_1} + T_{\boldsymbol{i}'}X_{k_1} = (T_{\boldsymbol{i}} + T_{\boldsymbol{i}'})X_{k_1}.$$

Similarly as in the case 1, we have $(T_{\boldsymbol{i}} + T_{\boldsymbol{i}'})X_{k_1} = 0$. Hence, $T_{\boldsymbol{i}} + T_{\boldsymbol{i}'} = 0$. Therefore, $T_{\boldsymbol{i}} = T_{\boldsymbol{i}'} = 0$.

Claim 2. $U \in A$.

By Claim 1, we have

$$U = S + T_1 + T_2 \quad (T_1 \in B_1',\ T_2 \in B_2').$$

By assumption,

$$UX_{k_1} - X_{k_1}U = (S + T_1)X_{k_1} - X_{k_1}(S + T_1) + T_2X_{k_1} - X_{k_1}T_2$$

belongs to B_1. Hence, we have $T_2X_{k_1} = X_{k_1}T_2 = 0$. Therefore, $T_2 = 0$. Similarly, the element

$$UY_{k_2} - Y_{k_2}U = SY_{k_2} - Y_{k_2}S + T_1Y_{k_2} - Y_{k_2}T_1$$

belongs to B_2. Hence, we have $T_1Y_{k_2} = Y_{k_2}T_1 = 0$. Therefore, $T_1 = 0$. Thus, we have $U = S \in A$.

Claim 3. $U \in \mathbb{Z} \cdot Z$.

By Claim 2, we have

$$U = \sum_{k=0}^{n} a_k Z^k \quad (a_k \in \mathbb{Z}).$$

Since U is a Lie element, $a_0 = 0$ and each homogeneous component a_kZ^k ($k \geqslant 1$) is a Lie element. But if $k \geqslant 2$, Z^k is not a Lie element. (This is a direct consequence of the Dynkin–Specht–Wever Theorem (cf., e.g., [10, Th. 5.17]).) Therefore, $U = a_1Z$ and the proof is completed. □

Remark. It seems plausible that Lemma 6 holds also in the case that either $g_1 = 1$ or $g_2 = 1$.

5. The rank of $\mathrm{gr}^m(\Gamma_g^n)$

(5.1) In this section, we shall give a lower bound for the rank of the module $\mathrm{gr}^m(\Gamma_g)$ for $m \geqslant 1$ and $g \geqslant 3$.

For $g = 0, 1$, the rank of $\mathrm{gr}^m(\Gamma_g^n)$ $(m \geqslant 1)$ is completely determined as follows. We first remark that in the case of $g = 0$, by definition,

$$\pi_{0,n}(2m-1) = \pi_{0,n}(2m) = m\text{th term of the lower central series of } \pi_{0,n}$$

for $m \geqslant 1$. Thus, we have

$$\Gamma_0^n[2m-1] = \Gamma_0^n[2m],$$

so that $\mathrm{gr}^{2m-1}(\Gamma_0^n) = \{0\}$.

Proposition 2. *Assume that $n \geqslant 2$.*

(i) $\operatorname{rank} \mathrm{gr}^m(\Gamma_g^n) = \operatorname{rank} \mathrm{gr}^m(\Gamma_g^{n-1}) + \operatorname{rank} \mathrm{gr}^m(\pi_{g,n-1})$.

(ii) $\operatorname{rank} \mathrm{gr}^{2m}(\Gamma_0^n) = (1/m)\sum_{i=1}^{n-2}\sum_{d|m}\mu(d)i^{m/d}$.

(iii) $\operatorname{rank} \mathrm{gr}^m(\Gamma_1^n) = (1/m)\sum_{r=1}^{n-1}\sum_{d|m}\mu(d)\{(1+\sqrt{r})^{m/d} + (1-\sqrt{r})^{m/d}\}$.

Proof. Recall that we have an exact sequence

$$1 \to \pi_{g,n-1}(m) \to \Gamma_g^n[m] \to \Gamma_g^{n-1}[m] \to 1$$

for $n \geqslant 2$ (cf. [2]). Statement (i) follows from this. By using that $\Gamma_0^3 = \{1\}$ and $\Gamma_1^1[1] = \{1\}$, (ii) and (iii) will be obtained from (i) and Theorem WL. $\square$

Let us consider the case that $(g_2, n_2) = (1, 1)$ in Theorem A. Then, we have injective homomorphisms

$$\mathrm{gr}^m(\Gamma_g^n) \to \mathrm{gr}^m(\Gamma_{g+1}^{n-1}).$$

Composing these successively, we have an injective homomorphism

$$\mathrm{gr}^m(\Gamma_1^{g-1}) \to \mathrm{gr}^m(\Gamma_g).$$

Thus, by Proposition 2(iii), we have the following lower bound for the rank of the module $\mathrm{gr}^m(\Gamma_g)$.

Theorem B. *For $m \geqslant 1$ and $g \geqslant 3$, we have*

$$\operatorname{rank} \mathrm{gr}^m(\Gamma_g) \geqslant \frac{1}{m}\sum_{r=1}^{g-2}\sum_{d|m}\mu(d)\bigl\{(1+\sqrt{r})^{m/d} + (1-\sqrt{r})^{m/d}\bigr\}.$$

Remark. The right hand side of the above inequality is a polynomial in g with rational coefficients. Its term of highest degree is

$$\begin{cases} \frac{1}{k(k+1)} g^{k+1} & (m = 2k), \\ \frac{2}{k+1} g^{k+1} & (m = 2k+1). \end{cases}$$

(5.2) In the case that $g = 0, 1$, the central filtration $\{\Gamma_g^n[m]\}_{m\geqslant 1}$ of $\Gamma_g^n[1]$ (essentially) coincides with the lower central filtration of $\Gamma_g^n[1]$. This seems to be more or less well known, but does not seem to explicitly appear in the literature. Moreover, in the case that $g \geqslant 2$, the filtration $\{\Gamma_g^n[m]\}_{m\geqslant 1}$ does not coincide with the lower central filtration (cf. [12]). Therefore, we shall give the proof for the convenience of the reader.

Proposition 3.
(i) *For $g = 0$, the subgroup $\Gamma_0^n[2m]$ is the mth term of the lower central series of Γ_0^n.*
(ii) *For $g = 1$, the subgroup $\Gamma_1^n[m]$ is the mth term of the lower central series of $\Gamma_1^n[1]$.*

For the proof, we need the following:

Lemma 7. *Let*

$$1 \to \pi_{g,n} \to G \to H \to 1$$

be an exact sequence of groups and $\{G_m\}_{m\geqslant 1}$ be a central series of G. Assume the following conditions are satisfied:
(i) *This sequence induces exact sequences*

$$1 \to \pi_{g,n}(m) \xrightarrow{d_m} G_m \to H(m) \to 1$$

for all $m \geqslant 1$, $\{H(m)\}_{m\geqslant 1}$ being the lower central series of H.
(ii) *The elements $d_2(z_j)$ $(1 \leqslant j \leqslant n)$ are all contained in $G(2)(=[G,G])$, the commutator subgroup of G.*

Then, $\{G_m\}_{m\geqslant 1}$ coincides with the lower central series of G, i.e., $G_m = G(m)$.

The proof is straightforward by induction on m.

Proof of Proposition 3. We apply Lemma 7 to the exact sequence

$$1 \to \pi_{g,n-1}(m) \to \Gamma_g^n[m] \to \Gamma_g^{n-1}[m] \to 1.$$

We first consider the case of $g = 0$. In the case of $n = 4$, this gives

$$\pi_{0,3}(2m) \simeq \Gamma_0^4[2m],$$

which settles the proposition in this case. By Lemma 7, the proposition follows by induction on n.

Let us consider the case of $g = 1$. Since $\Gamma_1^1[1] = \{1\}$, for $n = 2$, the above exact sequence gives

$$\pi_{1,1}(m) \simeq \Gamma_1^2[m],$$

which settles the proposition for $n = 2$. By a result of Birman [4], the condition (ii) of Lemma 7 is satisfied. Therefore, Lemma 7 can be applied and the proposition follows by induction on n. $\square$

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